Dot Products

Definition: The dot product of vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$ is the scalar in $\mathbb{R}$ defined by

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n \tag{1}$$

Example 1: Calculate $\mathbf{x} \cdot \mathbf{y}$ where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \tag{2}$$

$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$

$= (1)(-1) + (2)(0) + (-1)(-2) + (-3)(1) = -1 + 0 + 2 - 3 = \boxed{-2}$

Theorem 1 (Poole 1.2ad): For any vectors $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$ we have

1. $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$
2. $\mathbf{0} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{0} = 0$
3. $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$

Example 2: Justify theorem 1 in $\mathbb{R}^2$.

1) $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 = y_1 x_1 + y_2 x_2 = \vec{y} \cdot \vec{x}$

2) $\vec{0} \cdot \vec{x} = 0(x_1) + 0(x_2) = 0$ $\quad \vec{x} \cdot \vec{0} = 0$ by part one

3) $\vec{x} \cdot \vec{x} = x_1^2 + x_2^2 \geq 0$

a) Suppose $\vec{x} = \vec{0}$ Then $\vec{x} \cdot \vec{x} = 0$ by part 2

b) Suppose $\vec{x} \cdot \vec{x} = 0$, then $x_1^2 + x_2^2 = 0$

Then $x_1 = x_2 = 0$ and $\vec{x} = \vec{0}$.

Theorem 2 (Poole 1.2bc): For any vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^n$ and scalar c in $\mathbb{R}$ we have that

1. $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$
2. $\mathbf{x} \cdot (c\mathbf{y}) = c(\mathbf{x} \cdot \mathbf{y})$

Example 3: Justify theorem 2 in $\mathbb{R}^2$.

1) $x \cdot (\vec{y} + \vec{z}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} y_1 + z_1 \\ y_2 + z_2 \end{bmatrix} = x_1(y_1 + z_1) + x_2(y_2 + z_2)$

$= \underline{x_1 y_1} + \underline{x_1 z_1} + \underline{x_2 y_2} + \underline{x_2 z_2} = (x_1 y_1 + x_2 y_2) + (x_1 z_1 + x_2 z_2)$

$= \vec{x} \cdot \vec{y} + \vec{x} \cdot \vec{z}$

2) $\vec{x} \cdot (c\vec{y}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} cy_1 \\ cy_2 \end{bmatrix} = x_1(cy_1) + x_2(cy_2)$

$= c(x_1 y_1 + x_2 y_2)$

$= c(\vec{x} \cdot \vec{y})$

Example 4: Use theorems 1 and 2 to show that for any vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in $\mathbb{R}^n$ and scalars c, d in $\mathbb{R}$

$$(c\mathbf{x} + d\mathbf{y}) \cdot \mathbf{z} = c(\mathbf{x} \cdot \mathbf{z}) + d(\mathbf{y} \cdot \mathbf{z}) \tag{3}$$

$(c\vec{x} + d\vec{y}) \cdot \vec{z} = \vec{z} \cdot (c\vec{x} + d\vec{y})$ (theorem 1)

$= \vec{z} \cdot (c\vec{x}) + \vec{z} \cdot d\vec{y}$ (theorem 2)

$= c(\vec{z} \cdot \vec{x}) + d(\vec{z} \cdot \vec{y})$ (theorem 2)

$= c(\vec{x} \cdot \vec{z}) + d(\vec{y} \cdot \vec{z})$ (theorem 1)